Journal of Applied Mechanics and Technical Physics, Vol. 44, No. 3, pp. 317–323, 2003

EXACT SOLUTIONS OF THE HYDRODYNAMIC EQUATIONS DERIVED FROM PARTIALLY INVARIANT SOLUTIONS

V. V. Pukhnachev

UDC 532.511+517.9

The paper proposes a heuristic approach to constructing exact solutions of the hydrodynamic equations based on the specificity of these equations. A number of systems of hydrodynamic equations possess the following structure: they contain a "reduced" system of n equations and an additional equation for an "extra" function w. In this case, the "reduced" system, in which $w = 0$, admits a Lie group G. Taking a certain partially invariant solution of the "reduced" system with respect to this group as a "seed" solution, we can find a solution of the entire system, in which the functional dependence of the invariant part of the "seed" solution on the invariants of the group G has the previous form. Implementation of the algorithm proposed is exemplified by constructing new exact solutions of the equations of rotationally symmetric motion of an ideal incompressible liquid and the equations of concentrational convection in a plane boundary layer and thermal convection in a rotating layer of a viscous liquid.

Key words: *hydrodynamic equations, partially invariant solutions.*

1. Extension of the Set of Exact Solutions of the Hydrodynamic Equations. Many systems of hydrodynamic equations have the following specific structure:

$$L_1(u_1,\dots,u_n)+w=0,\quad L_k(u_1,\dots,u_n)=0,\ k=2,\dots,n,\quad \Lambda(u_1,\dots,u_n,w)=0,\tag{1.1}$$

and $\Lambda(u_1,\dots,u_n,0)=0$ for any $u=(u_1,\dots,u_n)\in R^n$. Here $L_1,\dots,L_n$, and Λ are differential operators (generally, nonlinear) acting on the variables $x=(x_1,\dots,x_m)\in R^m$.

Along with Eq. (1.1), we consider the "reduced" system

$$L_j(u_1,\dots,u_n)=0,\qquad j=1,\dots,n.\tag{1.2}$$

We assume that system (1.2) admits a local Lie group G acting in a space R^{m+n} and there is a partially invariant solution of system (1.2) with respect to the group G of the form

$$u_i=\varphi_i(I_1,\dots,I_l),\quad i=1,\dots,k<n,\qquad u_j=\psi_j(x_1,\dots,x_m),\quad j=k+1,\dots,n,\tag{1.3}$$

where $I_1,\dots,I_l$ ($l<m$) are the invariants of the group G that depend only on $x_1,\dots,x_m$. (Definition and procedure of constructing partially invariant solutions of a system of differential equations are given in [1].)

We substitute the expressions $u_i=\Phi_i(I_1,\dots,I_l)$, $u_j=\Psi_j(x_1,\dots,x_m)$ (Φ_i and Ψ_j are new *a priori* unknown functions) into system (1.1). Since the function w is explicitly expressed from the first equation of this system in terms of Φ_i and Ψ_j, we in fact obtain n equations that relate new desired functions. These equations should be supplemented with the differential relations expressing the invariance of the functions Φ_i with respect to the group G.

The overdetermined system of equations obtained is *a priori* compatible because it has a solution in which u_i and u_j are represented in the form of (1.3) and $w=0$. However, this system can also have "nontrivial" solutions. Below, we give three examples of such situations.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090.
Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 44, No. 3, pp. 18–25, May–June, 2003. Original article submitted November 21, 2002.

2. Rotationally Symmetric Motion of an Ideal Incompressible Liquid. In this section, u, v, and w are the projections of the velocity vector onto the r, θ, and z cylindrical coordinate axes, respectively, p is the liquid pressure, and ρ is the density. Instead of v, it is convenient to introduce a new desired function $\Omega = (rv)^2$, which is the squared rotational velocity circulation. In the adopted notation, the system of equations of rotationally symmetric motion for an ideal incompressible liquid is written as

$$\begin{gathered} r^3(u_t + uu_r + wu_z + \rho^{-1}p_r) - \Omega = 0, \\ w_t + uw_r + ww_z + \rho^{-1}p_z = 0, \qquad u_r + r^{-1}u + w_z = 0, \qquad \Omega_t + u\Omega_r + w\Omega_z = 0. \end{gathered} \tag{2.1}$$

The corresponding "reduced" system is formed by the first three equations in system (2.1):

$$\begin{gathered} u_t + uu_r + wu_z + \rho^{-1}p_r = 0, \\ w_t + uw_r + ww_z + \rho^{-1}p_z = 0, \qquad u_r + r^{-1}u + w_z = 0, \end{gathered} \tag{2.2}$$

and Ω plays the role of an "extra" function in Eq. (2.1). System (2.2) admits a three-parameter group G, whose Lie algebra is formed by operators $\{\partial_z, t\partial_z + \partial_w, \partial_p\}$. The complete set of functionally independent invariants of the group G can be chosen in the form

$$I_1 = r, \qquad I_2 = t, \qquad I_3 = u.$$

Since the rank of the set of the functions I_1, I_2, and I_3 with respect to the variables u, w, and p is zero, i.e., smaller than the number of desired functions in Eq. (2.2), invariant solutions of this system with respect to the group G do not exist. However, one can seek partially invariant solutions of system (2.2) with respect to the specified group, assuming that $u = u(r, t)$. Then, it follows from the third equation of system (2.2) that

$$w = \psi(r, t)z + \varphi(r, t),$$

where

$$\psi = -u_r - r^{-1}u. \tag{2.3}$$

Substitution of the expressions for u and w into the second equation of system (2.2) allows integration of this equation over z, which yields the following representation of the function p

$$-p/\rho = (\psi_t + u\psi_r + \psi^2)z^2/2 + (\varphi_t + u\varphi_r + \varphi\psi)z + \chi,$$

where $\chi(r, t)$ is the new desired function. Substituting the representations of u, w, and p into the first equation of system (2.2), we find that the left side of the resulting relation is a quadratic trinomial in z. Equating the coefficients of this trinomial to zero, we obtain the equations

$$(\psi_t + u\psi_r + \psi^2)_r = 0, \qquad (\varphi_t + u\varphi_r + \varphi\psi)_r = 0, \qquad u_t + uu_r + \chi_r = 0.$$

Along with Eq. (2.3), these equations form a closed system for the functions u, ψ, φ, and χ of the variables r and t that relates only the invariants of the group G. The number of independent variables in the system obtained determines the rank of the studied partially invariant solution, which is equal to two. In this case, the defect of the partially invariant solution [number of noninvariant desired functions w and p in the initial system (2.2)] is also equal to two.

Let us show how to construct a new solution of the complete system (2.1) using the partially invariant solution of the "reduced" system (2.2). According to the procedure described in Sec. 1, we assume that

$$u = U(r, t). \tag{2.4}$$

As a result, the second and third equations of system (2.1) are integrable over z. Integration over z yields

$$w = \Psi(r, t)z + \Phi(r, t); \tag{2.5}$$

$$-p/\rho = (\Psi_t + U\Psi_r + \Psi^2)z^2/2 + (\Phi_t + U\Phi_r + \Phi\Psi)z + \mathrm{X}, \tag{2.6}$$

where Φ, X, and Ψ are the desired functions r and t. In this case,

$$\Psi = -U_r - r^{-1}U. \tag{2.7}$$

Substitution of expressions (2.4)–(2.6) for u, w, and p into the first equation of system (2.1) yields the function Ω written as

$$\Omega = -r^3(\lambda z^2/2 + \mu z + \nu), \tag{2.8}$$

where

$$\lambda = (\Psi_t + U\Psi_r + \Psi^2)_r; \tag{2.9}$$

$$\mu = (\Phi_t + U\Phi_r + \Phi\Psi)_r; \tag{2.10}$$

$$\nu = \mathrm{X}_r - U_t - UU_r. \tag{2.11}$$

We substitute expression (2.8) for Ω into the last equation of system (2.1) and equate the coefficients of the quadratic trinomial in z to zero. As a result, we obtain three more equations

$$\lambda_t + U\lambda_r + (2\Psi + 3r^{-1}U)\lambda = 0; \tag{2.12}$$

$$\mu_t + U\mu_r + (\Psi + 3r^{-1}U)\mu + \Phi\lambda = 0; \tag{2.13}$$

$$\nu_t + U\nu_r + 3r^{-1}U\nu = 0, \tag{2.14}$$

which, along with Eqs. (2.7) and (2.9)–(2.11), form a closed system for seven functions (U, Ψ, Φ, X, λ, μ, and ν) of the variables r and t. This system has a recurrent structure, which makes its analysis much simpler. Indeed, Eqs. (2.7), (2.9), and (2.12) are related only by the functions U, Ψ, and λ. After these functions are determined, the functions Φ and μ are found from system (2.10) and (2.13), and the functions ν and X are obtained from Eqs. (2.14) and (2.11) solved in series. In this case, subsystems (2.10), (2.13), (2.14), and (2.11) are linear in the desired functions Φ, μ, ν, and X, respectively. Without dwelling on the solution of the indicated subsystems, we shall focus on the nonlinear system (2.7), (2.9), and (2.12) and showing that it can be reduced to a more standard form.

In system (2.7), (2.9), and (2.12), we convert from r to a new spatial variable — the Lagrangian coordinate ξ defined by the relations

$$\frac{dr}{dt} = U(r,t) \quad \text{for} \quad t > 0, \qquad r = a(\xi) \quad \text{for} \quad t = 0. \tag{2.15}$$

Here $a(\xi)$ is a function that satisfies the conditions $a(\xi) \in C^2[\xi_1, \xi_2]$, $a(0) \geqslant 0$, and $a'(\xi) > 0$ for $\xi \in [\xi_1, \xi_2]$; in other respects, this is an arbitrary function (we shall handle this arbitrariness later). We introduce the notation $l(\xi,t) = \lambda[r(\xi,t),t]$ and $f(\xi,t) = \Psi[r(\xi,t),t]$. Elimination of the function Ψ from Eqs. (2.12) and (2.7) leads to the equality

$$\lambda_t + U\lambda_r + (-2U_r + U)\lambda = 0.$$

Converting to the new variables in the above equation and taking into account that $\lambda_t + U\lambda_r = l_t$, $U_r = r_{\xi t}/r_\xi$, we obtain the relation

$$l_t/l + r_t/r - 2r_{\xi t}/r_\xi = 0,$$

which is integrated over t as

$$rl/r_\xi^2 = \sigma(\xi),$$

where σ is an arbitrary function of ξ. Omitting the insignificant case of $\sigma = 0$, we reduce consideration of the neighborhood of each point where the function σ maintains sign to the case of $\sigma = 1$ or $\sigma = -1$ by converting to a new variable $\tilde{\xi}$ defined by the relations $d\tilde{\xi} = \sigma^{1/2}(\xi)\,d\xi$ or $d\tilde{\xi} = [-\sigma(\xi)]^{1/2}\,d\xi$. In this case, only the right side of the initial condition (2.15) changes but the new function $\tilde{a}(\tilde{\xi})$ has the same properties as $a(\xi)$ and the previous notation (ξ and a instead of $\tilde{\xi}$ and $\tilde{a}$) can be retained.

Thus, we obtained the equality $l = \sigma r^{-1} r_\xi^2$, in which the quantity σ takes a value of 1 or -1. Converting to Lagrangian coordinates in (2.9) and substituting the expression of l in terms of r and r_ξ into the equality obtained, we have $(f_t + f^2)_\xi = \sigma r^{-1} r_\xi^3$. Another equation relating r and f is obtained by converting to new variables in equality (2.7): $rr_\xi f = -(rr_t)_\xi$. Introduction of the new desired function $y(\xi,t) = r^2/8$ simplifies the resulting system of equations for r and f, which takes the final form

$$y_{\xi t} = -y_\xi f; \tag{2.16}$$

$$(f_t + f^2)_\xi = \sigma y^{-2} y_\xi^3. \tag{2.17}$$

System (2.16) and (2.17) should be considered be hyperbolic although it was derived from the equations describing the motion of an incompressible liquid. As is known, the system of Euler equations for an incompressible liquid is compound: it has both real and complex characteristics. In the reduction of these equations, their hyperbolic component is separated from the elliptic component, which facilitates analysis of the initial-boundary-value problem. (We note that other hyperbolic models of motion for an incompressible liquid were studied in [2, 3].)

Elimination of f from system (2.16) and (2.17) yields the following forth-order hyperbolic equation for the function $y(\xi, t)$:

$$\Big(\frac{y_{\xi\xi}}{y_\xi}\Big)_{tt} = \Big[\Big(\frac{y_{\xi t}}{y_\xi}\Big)^2\Big]_\xi - \sigma\,\frac{y_\xi^3}{y^2}. \tag{2.18}$$

The natural initial-boundary-value problem for (2.18) is the following:

$$y(\xi, 0) = y_0(\xi), \qquad y_t(\xi, 0) = y_1(\xi), \qquad \xi_1 \leqslant \xi \leqslant \xi_2; \tag{2.19}$$

$$y(\xi_1, t) = c_1, \qquad y(\xi_2, t) = c_2, \qquad t > 0. \tag{2.20}$$

Here ξ_1, ξ_2, and $c_2 > c_1 > 0$ are specified constants and $y_0(\xi) > 0$ and $y_1(\xi)$ are specified functions.

Below, we assume that $y_0 \in C^2[\xi_1, \xi_2]$ and $y_1 \in C^1[\xi_1, \xi_2]$; moreover, the compatibility conditions $y_0(\xi_i) = c_i$ and $y_1(\xi_i) = 0$ $(i = 1, 2)$ and the monotonicity condition $y_0'(\xi) > 0$ for $\xi \in [\xi_1, \xi_2]$ are satisfied. Conditions (2.19) and (2.20) have a clear physical meaning. As follows from (2.19) and the relations $r = 2(2y)^{1/2}$ and $r_t = U$, the equalities $r = 2[2y_0(\xi)]^{1/2}$ and $rU = 4y_1(\xi)$ are satisfied for $t = 0$. Excluding the parameter ξ from these equalities, we obtain the initial distribution of the radial velocity component $U(r, 0) = U_0(r)$ for $c_1^2/8 \leqslant r \leqslant c_2^2/8$. Equalities (2.20) mean that the nonpenetration conditions $r_{i,t} = U(r_i, t) = 0$ $(i = 1, 2)$ are satisfied for $r_i = c_i^2/8 = \text{const}$. This allows the examined solution of the Euler equations to be treated as the one describing the motion in a cylindrical layer with impermeable walls that arises from the specified initial state. This solution generalizes the well-known Aristov's solution [4] in two directions: first, unsteady motion is considered and, second, more importantly, according to (2.8), the function Ω (squared rotational velocity circulation), is a full quadratic trinomial in z, whereas in [4], Ω is proportional to z^2.

Omitting a detailed analysis of problem (2.18)–(2.20), we only note that if the above-formulated conditions of smoothness, compatibility, and monotonicity of the initial data of the problem are satisfied, the existence and uniqueness theorem for the classical solution on a small time interval is valid for this problem. It is rather difficult to obtain sufficient conditions of its solvability on an arbitrary interval $[0, T]$. Moreover, the possibility that the solution of problem (2.18)–(2.20) for certain initial data $y_0(\xi)$ and $y_1(\xi)$ decays over a finite time cannot be *a priori* ruled out. It would be of interest to conduct numerical experiments to study the behavior of the solution of the problem for larger t.

3. Concentrational Convection in a Plane Boundary Layer. In this section, u and v denote the projections of the velocity onto the x and y axes of Cartesian coordinates, respectively, p is the difference between the liquid pressure and hydrostatic pressure, c is the concentration of the inactive admixture, ν is the kinematic viscosity of the liquid, D is the diffusion coefficient of the admixture, and g is the acceleration of gravity acting in the negative x direction. The dependence of the liquid density ρ on the admixture concentration is considered linear: $\rho = \rho_0(1 - \alpha c)$. It is assumed that the parameters ν, D, g, and ρ_0 are positive constants. The parameter α is also assumed to be constant, but its sign can be arbitrary.

The equations of unsteady convection in a plane boundary layer are written as

$$\begin{gathered} u_t + uu_x + vu_y = -\rho_0^{-1} p_x + \nu u_{yy} + g\alpha c, \\ p_y = 0, \qquad u_x + v_y = 0, \qquad c_t + uc_x + vc_y = Dc_{yy}. \end{gathered} \tag{3.1}$$

The "reduced" system corresponds to the motion of a homogeneous liquid. Setting $c = 0$ in (3.1), we obtain the classical equations of an unsteady plane boundary layer:

$$u_t + uu_x + vu_y = -\rho_0^{-1} p_x + \nu u_{yy}, \qquad p_y = 0, \qquad u_x + v_y = 0. \tag{3.2}$$

System (3.2) admits the group G with the basis operators $\{\partial_t, \partial_y, x\partial_x + u\partial_u + 2p\partial_p\}$. The basis of the invariants of the group G is as follows: $I_1 = x^{-1}u$, $I_2 = v$, and $I_3 = x^{-2}p$. Since it does not include invariants that do not contain the desired functions, system (3.2) does not have regular partially invariant solutions with respect to the group G. Ondich [5] constructed an irregular partially invariant solution of (3.2) on the group G by relating the invariants I_1 and I_2 by an equality, i.e., by setting

$$v = x^{-1}u. \tag{3.3}$$

Substitution of Eq. (3.3) into the last equation in system (3.2) yields the following representation of the function u:

$$u = \psi(x\mathrm{e}^{-y}, t). \tag{3.4}$$

Substituting Eq. (3.3) and (3.4) into the first equation in system (3.2) and denoting $\xi = x\mathrm{e}^{-y}$, we obtain the following relationship between the functions ψ and p:

$$\psi_t = -\rho_0^{-1}p_x + \nu(\xi^2\psi_{\xi\xi} + \xi\psi_\xi).$$

Eliminating the function p from the last equality by differentiating it with respect to y, we arrive at the following equation for the function ψ:

$$\psi_{\xi t} = \nu(\xi^2\psi_{\xi\xi\xi} + 3\xi\psi_{\xi\xi} + \psi_\xi).$$

This is a linear equation and it admits separation of variables, which allowed us to find a number of exact solutions of this equation [5]. A solution of the general Cauchy problem for this equation can be constructed by applying the Mellin transform.

We now consider the full system (3.1) and will seek its solution in which the relation between the velocity components has the previous form of (3.3) and the representation

$$u = \Psi(x\mathrm{e}^{-y}, t) \tag{3.5}$$

contains a function $\Psi(\xi, t)$ that generally does not coincide with $\psi(\xi, t)$. Substitution of Eq. (3.5) and (3.3) into the first equation of system (3.1) yields the equality

$$\Psi_t = -\rho_0^{-1}p_x + \nu(\xi^2\Psi_{\xi\xi} + \xi\Psi_\xi) + g\alpha c,$$

which, in turn, yields the representation

$$c = (g\alpha)^{-1}[\rho_0^{-1}p_x + q(\xi, t)], \tag{3.6}$$

where the function q is linked to Ψ by the relation

$$\Psi_t = \nu(\xi^2\Psi_{\xi\xi} + \xi\Psi_\xi) + q. \tag{3.7}$$

Substituting Eqs. (3.3), (3.5), and (3.6) into the last equation in system (3.1), we obtain the closing relation between the functions Ψ, q, and p:

$$q_t + \rho_0^{-1}(p_{xt} + p_{xx}\Psi) = D(\xi^2 q_{\xi\xi} + \xi q_\xi). \tag{3.8}$$

So far, we have not impose restrictions on the function $p(x, t)$. Now we choose this function such that the left side of Eq. (3.8) is a function of the variables ξ and t. For this, it suffices to set

$$p = \rho_0[\beta(t) + \gamma(t)x + \delta(t)x^2], \tag{3.9}$$

where β, γ, and δ are arbitrary functions of t. Substitution of (3.9) into (3.8) yields

$$q_t + \dot{\gamma}(t) + 2\delta(t)\Psi = D(\xi^2 q_{\xi\xi} + \xi q_\xi). \tag{3.10}$$

Thus, a new solution of the equations of unsteady concentrational convection in a plane boundary layer was constructed on the basis of the well-known partially invariant solution of the equations of a dynamic boundary layer. In this solution, the pressure expressed by formula (3.9) depends quadratically on x, whereas in the solution given in [5], this dependence is not more than linear. The velocity fields and concentrations in the solution constructed are specified by equalities (3.3), (3.5), and (3.6), where the functions Ψ and q satisfy system (3.7), (3.10). Being linear, this system admits separation of variables. For this system, the natural formulation of the initial-boundary-value problem is given by

$$\Psi(\xi, 0) = \Psi_0(\xi), \qquad q(\xi, 0) = q_0(\xi), \qquad \xi > 0,$$

$$|\Psi| < \infty, \quad |q| < \infty \qquad \text{for} \quad \xi \to 0 \quad \text{and} \quad \xi \to \infty.$$

Without discussing the resolvability conditions for the given problem, we only note one characteristic feature of the solution of system (3.1), which is also inherent in the solution of system (3.2) derived by Ondich: because of the specific dependence of the function u on y given by formula (3.5) or (3.4), the attachment condition $u = 0$ cannot be satisfied for $y = 0$ (and for any $y = \text{const}$). This prevents the use of this solution to describe concentrational convection in the vicinity of a rigid vertical wall. However, this solution can be used in some other cases, for example, in problems of mixing layers or submerged jets. In addition, this solution can find an application as a test for numerical integration of system (3.1). Besides nonlinearity and strong degeneration, this system implicitly contains the small parameter $\nu^{-1}D$; i.e., it is, in fact, a singularly perturbed system (for real liquids, the ratio of D to ν does not exceed 10^{-3}). In such situations, the existence of meaningful exact solutions is of great importance.

4. Thermal Convection in a Rotating Layer of a Viscous Liquid. In this section, the initial equations are as follows:

$$\begin{gathered} u_t + uu_r + wu_z - 2\omega v - v^2/r = -p_r/\rho_0 + \nu(u_{rr} + u_r/r - u/r^2 + u_{zz}) - \omega^2\beta rT, \\ v_t + uv_r + wv_z + 2\omega u + uv/r = \nu(v_{rr} + v_r/r - v/r^2 + v_{zz}), \\ w_t + uw_r + ww_z = -p_z/\rho_0 + \nu(w_{rr} + w_r/r + w_{zz}), \\ u_r + u/r + w_z = 0, \qquad T_t + uT_r + wT_z = \chi(T_{rr} + T_r/r + T_{zz}). \end{gathered} \tag{4.1}$$

Equations (4.1) describe the thermal convection of a viscous incompressible liquid assuming rotationally symmetric motion. They are written in a coordinate system rotating at constant angular velocity ω relative to the initial inertial system. The rotation axis coincides with the z axis of the cylindrical coordinate system (r, θ, z). We use u, v, and w to denote the radial and axial velocity components and the deviation of the rotational velocity component from the velocity of rigid-body rotation ωr, respectively. The quantity p characterizes the deviation of the pressure from the equilibrium value $\rho_0\omega^2 r^2/2$, and the quantity T is the temperature deviation from a certain average value. The positive parameters ρ_0, ν, β, and χ have the following physical meaning: ρ_0 is the liquid density at a constant temperature ($T = 0$), ν is the kinematic viscosity coefficient, β is the volumetric thermal-expansion coefficient of the liquid, and χ is the thermal conductivity.

A "reduced" system is derived from Eq. (4.1) if we set $T = 0$ in the first equation and discard the last equation. This system describes the rotationally symmetric motion of an isothermally viscous incompressible liquid and admits solutions of the form

$$u = rf(z,t), \qquad v = rg(z,t), \qquad w = w(z,t), \qquad p = K(t)r^2/2 + h(z,t), \tag{4.2}$$

which are generalizations of the well-known von Kármán solution. The theoretical-group nature of solution (4.2) is rather nontrivial: similarly to the von Kármán solution, it is an invariant solution of a partially invariant submodel of the full (three-dimensional) Navier–Stokes equations. This submodel is generated by the four-parameter group formed by two translations along the x_1 and x_2 Cartesian coordinate axes and two Galilean translations along the same axes [6].

Applying the above-described method to (4.2), one can obtain a solution of system (4.1) in the form

$$p = K(t)r^2/2 + A\rho_0\beta\omega^2((r^2/2)\ln(r/a) - r^2/4) + h(z,t), \qquad T = A\ln(r/a) + S(z,t),$$

where A and a are constants having dimensions of temperature and length, respectively; the velocity field preserves the form of (4.2); and the functions f, g, w, and h are determined simultaneously with the function S. The system for determining these functions is written as

$$\begin{gathered} f_t + wf_z - 2\omega g + f^2 - g^2 = -\rho_0^{-1}K + \nu f_{zz} - \omega^2\beta S, \qquad g_t + wg_z + 2\omega f + 2fg = \nu g_{zz}, \\ 2f + w_z = 0, \qquad S_t + wS_z + Af = \chi S_{zz}, \qquad w_t + ww_z = -\rho_0^{-1}h_z + \nu w_{zz}. \end{gathered} \tag{4.3}$$

The form of the solution obtained suggests its possible physical interpretation. The liquid fills the layer between rigid planes $z = \pm a$ rotating at angular velocity ω around the z axis. The attachment condition is satisfied on the planes. Drains or heat sources of constant linear density $-2\pi Ak$ (k is the of thermal conductivity of the liquid) are distributed along the rotation axis. The planes bounding the flow are heat insulated. At the initial time, the velocity distribution over the layer is specified by formulas (4.2). These conditions induce the following formulation of the initial-boundary-value problem for system (4.3):

$$f = g = w = 0, \quad S_z = 0 \qquad \text{for} \quad z = \pm a, \ t > 0; \tag{4.4}$$

$$f = -w_0'(z)/2, \quad g = g_0(z), \quad w = w_0(z), \quad S = S_0(z) \qquad \text{for} \quad |z| \leqslant a, \ t = 0, \tag{4.5}$$

where w_0, g_0, and S_0 are specified functions of z. If the natural smoothness and compatibility conditions imposed on the indicated functions are satisfied, problem (4.3)–(4.5) has a classical solution at least on a small time interval $[0, \tau]$. This solution is unique with accuracy to addition of an arbitrary function of t to h.

In conclusion, we note that for any governing parameters, system (4.3) has a trivial solution in which all the desired functions are zero. This solution describes the equilibrium of a uniformly rotating liquid in an infinite layer whose boundaries are rigid impermeable heat-insulated planes, and drains and heat sources with constant linear density are distributed along the rotation axes. It can be shown that for sufficiently large positive values of the parameter A, the trivial solution is unstable.

The results of this paper were reported at the 3rd International Conference "Symmetry and Differential Equations" (Krasnoyarsk, August 2002) and at Ovsyannikov's seminar. The author thanks L. V. Ovsyannikov and the seminar participants S. V. Golovin, S. V. Meleshko, and A. P. Chupakhin for their attention to the work and helpful discussions and thanks E. Yu. Meshcheryakova for assistance in preparing the manuscript of the paper.

REFERENCES

1. L. V. Ovsyannikov, *Group Analysis of Differential Equations,* Academic Press, New York (1982).
2. V. V. Pukhnachev, "New class of exact solutions of the Euler equations," *Dokl. Ross. Akad. Nauk*, **382**, No. 6, 777–780 (2002).
3. E. Yu. Meshcheryakova, "Exact solutions of equations rotationally symmetric motion of a ideal incompressible liquid," *J. Appl. Mech. Tech. Phys.*, **43**, No. 3, 397–405 (2002).
4. S. N. Aristov, "Steady cylindrical vortex in a viscous liquid," *Dokl. Ross. Akad. Nauk*, **377**, No. 4, 477–480 (1991).
5. J. Ondich, "A differential constrains approach to partial invariance," *Eur. J. Appl. Math.*, **6**, No. 4, 631–637 (1995).
6. S. V. Meleshko and V. V. Pukhnachev, "One class of partially invariant solutions of the Navier–Stokes equations," *J. Appl. Mech. Tech. Phys.*, **40**, No. 2, 208–216 (1999).